ANNALS OF MATHEMATICS.

VOL. II. AUGUST, 1886. NO. 4.

ON THE USE OF SOMOFF'S THEOREM FOR THE EVALUATION OF THE ELLIPTIC INTEGRAL OF THE THIRD SPECIES.

By MR. CHAS. H. KUMMELL, Washington, D. C.

The theorem which I call Somoff's, is a relation between elliptic integrals of the third species in successive steps in the scale of moduli and corresponding amplitudes and parametric angles. I had, however, independently discovered this theorem, and founded on it the methods which I am going to explain, before I saw Somoff's article in *Crelle's Journal*, Vol. XLVII, pp. 269–289, and since they are essentially different from Somoff's methods, and in some respects more convenient, it may be of some use to give them also.

If, according to the notation which I have used in previous articles,

$$\varphi_\gamma = \int_0^\phi \frac{d\varphi}{\sqrt{(1-\gamma^2\sin^2\varphi)}} = \int_0^\phi \frac{d\varphi}{\sqrt{(\beta^2+\gamma^2\cos^2\varphi)}} = \int_0^\phi \frac{d\varphi}{\Delta\varphi}, \tag{1}$$

then Jacobi's type form of the elliptic integral of the third species is

$$\begin{aligned} \Pi(\varphi_\gamma, \mu_\gamma) &= \int_0^\phi \frac{d\varphi}{\Delta\varphi}\cdot\frac{\gamma^2\sin^2\varphi}{1-\gamma^2\sin^2\mu\sin^2\varphi}\sin\mu\cos\mu\,\Delta\mu \\ &= \int_0^\phi \frac{d\varphi}{\Delta\varphi}\cdot\frac{1-\Delta^2\varphi}{\cos^2\mu+\sin^2\mu\,\Delta^2\varphi}\sin\mu\cos\mu\,\Delta\mu \\ &= \frac{\Delta\mu}{\tan\mu}[\Pi_1(\varphi,\gamma,-\gamma^2\sin^2\mu)-\varphi_\gamma], \end{aligned} \tag{2}$$

where Π_1 denotes Legendre's type for the third species, in which the parameter

$$n = -\gamma^2\sin^2\mu. \tag{3}$$

This integral vanishes for all values of μ which cause either $\sin\mu$, $\cos\mu$, or $\Delta\mu$

to vanish; viz.: if $\mu = 2m \lrcorner$,* or $= (2m' + 1) \lrcorner$, or $= [(2m + 1) \lrcorner_\gamma + (2m' + 1) \lrcorner_\beta i]_{-\gamma}$, or, in the simplest cases, if $\mu = 0$, or $= \lrcorner$, or $= (\lrcorner_\gamma + \lrcorner_\beta i)_{-\gamma}$.

The parametric angle μ cannot be real in all cases, and it is necessary to assume the more general form,

$$\begin{aligned} n &= -\gamma^2 \sin^2 (\mu_\gamma + \nu_\beta i)_{-\gamma} \\ &= -\left(\frac{\gamma}{1 - \Delta^2\mu \sin^2 \nu}\right)^2 [\sin \mu \, \Delta(\nu_\beta)_{-\beta} + i \cos \mu \, \Delta\mu \sin \nu \cos \nu]^2. \qquad (4) \end{aligned}$$

This is real if

1. $\sin \nu = 0$, then $n = -\gamma^2 \sin^2 \mu$; $\qquad (0 < -n < \gamma^2) \quad (4_1)$
2. $\sin \mu = 0$, then $n = -\gamma^2 \sin^2 (\nu_\beta i)_{-\gamma} = \gamma^2 \tan^2 \nu$; $\quad (-\infty < -n < 0) \quad (4_2)$
3. $\cos \nu = 0$, then $n = -\gamma^2 \sin^2 (\mu_\gamma + \lrcorner_\beta i)_{-\gamma} = -\dfrac{1}{\sin^2 \mu}$; $\qquad (1 < -n < \infty) \quad (4_3)$
4. $\cos \mu = 0$, then $n = -\gamma^2 \sin^2 (\lrcorner_\gamma + \nu_\beta i)_{-\gamma} = -\dfrac{\gamma^2}{\Delta^2(\nu_\beta)_{-\beta}}$ $\qquad (\gamma^2 < -n < 1) \quad (4_4)$

This completes the range of real values for parameter. There are, however, four other forms extending over the same ranges respectively, which are obtained from these by taking elliptic complements to μ or ν; thus,

1'. $n = -\gamma^2 \sin^2 (\lrcorner_\gamma + \mu_\gamma)_{-\gamma} = -\gamma^2 \dfrac{\cos^2 \mu}{\Delta^2 \mu}$, $\qquad (0 < -n < \gamma^2) \quad (4_1')$

2'. $n = -\gamma^2 \sin^2 (\lrcorner_\beta i + \nu_\beta i)_{-\gamma} = \gamma^2 \cot^2 \nu$, $\qquad (-\infty < -n < 0) \quad (4_2')$

3'. $n = -\gamma^2 \sin^2 (\lrcorner_\gamma + \mu_\gamma + \lrcorner_\beta i)_{-\gamma} = -\dfrac{\Delta^2 \mu}{\cos^2 \mu}$, $\qquad (1 < -n < \infty) \quad (4_3')$

4'. $n = -\gamma^2 \sin^2 (\lrcorner_\gamma + \lrcorner_\beta i + \nu_\beta i)_{-\gamma} = -\Delta^2 (\nu_\beta)_{-\beta}$. $\quad (\gamma^2 < -n < 1) \quad (4_4')$

These last four cases, which I give only for reference, are not essentially different from the first four, and they may replace each other respectively. By means of the theorem for the addition of parameters,

$$\begin{aligned} \Pi(\varphi_\gamma, \mu_\gamma + \nu_\beta i) = {} & \Pi(\varphi_\gamma, \mu_\gamma) + \Pi(\varphi_\gamma, \nu_\beta i) \\ & - \gamma^2 \varphi_\gamma \sin \mu \sin (\nu_\beta i)_{-\gamma} \sin (\mu_\gamma + \nu_\beta i)_{-\gamma} \\ & + \tfrac{1}{2} l \frac{1 + \gamma^2 \sin \mu \sin (\nu_\beta i)_{-\gamma} \sin \varphi \sin (\mu_\gamma + \nu_\beta i + \varphi_\gamma)_{-\gamma}}{1 - \gamma^2 \sin \mu \sin (\nu_\beta i)_{-\gamma} \sin \varphi \sin (\mu_\gamma + \nu_\beta i - \varphi_\gamma)_{-\gamma}}, \qquad (5) \end{aligned}$$

*See ANNALS OF MATHEMATICS, Vol. II, p. 38.

we can reduce all cases to the first two forms, viz.: —

1. the logarithmic integral,

$$\Pi(\varphi_\gamma, \mu_\gamma) = \int_0^\phi \sin \mu \cos \mu \Delta \mu \, \frac{d\varphi}{\Delta\varphi} \cdot \frac{\gamma^2 \sin^2 \varphi}{1 - \gamma^2 \sin^2 \mu \sin^2 \varphi},$$

2. the cyclometric integral,

$$\Pi(\varphi_\gamma, \nu_\beta i) = i \int_0^\phi \frac{\tan \nu \, \Delta(\nu_\beta)_{-\beta}}{\cos^2 \nu} \cdot \frac{d\varphi}{\Delta\varphi} \cdot \frac{\gamma^2 \sin^2 \varphi}{1 + \gamma^2 \tan^2 \nu \sin^2 \varphi};$$

and we have the following special formulæ of reduction: —

$$\Pi(\varphi_\gamma, \rfloor_\gamma + \mu_\gamma) = \Pi(\varphi_\gamma, \mu_\gamma) - \gamma^2 \varphi_\gamma \sin \mu \sin(\rfloor_\gamma + \mu_\gamma)_{-\gamma}$$
$$+ \tfrac{1}{2} l \frac{1 + \gamma^2 \sin \mu \sin \varphi \sin(\rfloor_\gamma + \mu_\gamma + \varphi_\gamma)_{-\gamma}}{1 - \gamma^2 \sin \mu \sin \varphi \sin(\rfloor_\gamma + \mu_\gamma - \varphi_\gamma)_{-\gamma}}$$
$$= \Pi(\varphi_\gamma, \mu_\gamma) - \gamma^2 \varphi_\gamma \sin \mu \frac{\cos \mu}{\Delta \mu} + \tfrac{1}{2} l \frac{\Delta(\mu_\gamma - \varphi_\gamma)_{-\gamma}}{\Delta(\mu_\gamma + \varphi_\gamma)_{-\gamma}}, \quad (6)$$

$$\Pi(\varphi_\gamma, \rfloor_\gamma + \nu_\beta i) = \Pi(\varphi_\gamma, \nu_\beta i) - \gamma^2 i \varphi_\gamma \frac{\tan \nu}{\Delta(\nu_\beta)_{-\beta}}$$
$$+ \tfrac{1}{2} l \frac{\Delta\varphi \cos \nu \, \Delta(\nu_\beta)_{-\beta} + i \sin \varphi \cos \varphi \sin \nu}{\Delta\varphi \cos \nu \, \Delta(\nu_\beta)_{-\beta} - i \sin \varphi \cos \varphi \sin \nu}$$
$$= \Pi(\varphi_\gamma, \nu_\beta i) - \gamma^2 i \varphi_\gamma \frac{\tan \nu}{\Delta(\nu_\beta)_{-\beta}}$$
$$+ i \operatorname{arc\,tan} \left(\frac{\sin \varphi \cos \varphi}{\Delta\varphi} \cdot \frac{\tan \nu}{\Delta(\nu_\beta)_{-\beta}} \right), \quad (7)$$

$$\Pi(\varphi_\gamma, \rfloor_\gamma + \mu_\gamma + \rfloor_\beta i) = \Pi(\varphi_\gamma, \mu_\gamma) - \varphi_\gamma \tan \mu \, \Delta \mu + \tfrac{1}{2} l \frac{\cos(\mu_\gamma - \varphi_\gamma)_{-\gamma}}{\cos(\mu_\gamma + \varphi_\gamma)_{-\gamma}}, \quad (8)$$

$$\Pi(\varphi_\gamma, \rfloor_\gamma + \rfloor_\beta i + \nu_\beta i) = \Pi(\varphi_\gamma, \nu_\beta i) - \varphi_\gamma i \tan \nu \, \Delta(\nu_\beta)_{-\beta}$$
$$+ i \operatorname{arc\,tan} [\tan \varphi \Delta\varphi \tan \nu \, \Delta(\nu_\beta)_{-\beta}]. \quad (9)$$

These formulæ suffice for the required reduction, and we may suppose given the simple logarithmic and cyclometric integral for evaluation.

Somoff's theorem I deduce as follows: —

Assume by Landen's transformation, ascending,

$$\sin \varphi = (1 + \beta') \sin \varphi' \frac{\cos \varphi'}{\Delta \varphi'} = (1 + \beta') \sin \varphi' \sin(\rfloor_{\gamma'} - \varphi'_{\gamma'})_{-\gamma'}, \quad (10)$$

$$\sin \mu = (1 + \beta') \sin \mu' \frac{\cos \mu'}{\Delta \mu'} = (1 + \beta') \sin \mu' \sin(\rfloor_{\gamma'} - \mu'_{\gamma'})_{-\gamma'}, \quad (11)$$

where $\beta', \gamma', \varphi', \mu'$ correspond to $\beta, \gamma, \varphi, \mu$ respectively on the next higher step of the modular scale. We have also

$$\gamma = \frac{1-\beta'}{1+\beta'}, \tag{12}$$

$$\Delta\mu = \frac{\Delta\mu' + \dfrac{\beta'}{\Delta\mu'}}{1+\beta'} = \frac{\Delta\mu' + \Delta(\lrcorner_{\gamma'} - \mu'_{\gamma'})_{-\gamma'}}{1+\beta'}, \tag{13}$$

$$\cos\mu = \frac{\Delta\mu' - \dfrac{\beta'}{\Delta\mu'}}{1-\beta'} = \frac{\Delta\mu' - \Delta(\lrcorner_{\gamma'} - \mu'_{\gamma'})_{-\gamma'}}{1-\beta'}, \tag{14}$$

$$\frac{d\varphi}{\Delta\varphi} = (1+\beta')\frac{d\varphi'}{\Delta\varphi'}; \tag{15}$$

whence

$$\Pi(\varphi_\gamma, \mu_\gamma) = \int_0^{\phi'} (1+\beta')\frac{d\varphi'}{\Delta\varphi'}(1+\beta')\sin\mu'\frac{\cos\mu'}{\Delta\mu'} \cdot \frac{\Delta^2\mu' - \dfrac{\beta'^2}{\Delta^2\mu'}}{1-\beta'^2} \cdot \frac{(1-\beta')^2\sin^2\varphi'\dfrac{\cos^2\varphi'}{\Delta^2\varphi'}}{1-\gamma'^4\sin^2\mu'\dfrac{\cos^2\mu'}{\Delta^2\mu'}\sin^2\varphi'\dfrac{\cos^2\varphi'}{\Delta^2\varphi'}}$$

$$= \int_0^{\phi'} \frac{d\varphi'}{\Delta\varphi'}\gamma'^2\sin^2\varphi'\sin\mu'\cos\mu'\,\Delta\mu' \cdot \frac{\left(\Delta^2\mu' - \dfrac{\beta'^2}{\Delta^2\mu'}\right)\cos^2\varphi'}{\Delta^2\mu'\Delta^2\varphi' - \gamma'^4\sin^2\mu'\sin^2\varphi'\cos^2\mu'\cos^2\varphi'}$$

$$= \int_0^{\phi'} \frac{d\varphi'}{\Delta\varphi'}\gamma'^2\sin^2\varphi'\sin\mu'\cos\mu'\,\Delta\mu' \cdot \left\{\frac{1}{1-\gamma'^2\sin^2\mu'\sin^2\varphi'} - \frac{\dfrac{\beta'^2}{\Delta^4\mu'}}{1-\gamma'^2\dfrac{\cos^2\mu'}{\Delta^2\mu'}\sin^2\varphi'}\right\}.$$

Since
$$\sin(\lrcorner_{\gamma'} - \mu'_{\gamma'})_{-\gamma'} = \frac{\cos\mu'}{\Delta\mu'},$$

$$\cos(\lrcorner_{\gamma'} - \mu'_{\gamma'})_{-\gamma'} = \frac{\beta'\sin\mu'}{\Delta\mu'},$$

$$\lrcorner(\lrcorner_{\gamma'} - \mu'_{\gamma'})_{-\gamma'} = \frac{\beta'}{\Delta\mu'};$$

we have
$$\sin(\lrcorner_{\gamma'} - \mu'_{\gamma'})_{-\gamma'}\cos(\lrcorner_{\gamma'} - \mu'_{\gamma'})_{-\gamma'}\,\Delta(\lrcorner_{\gamma'} - \mu'_{\gamma'})_{-\gamma'}$$

$$= \frac{\beta'^2\sin\mu'\cos\mu'}{\Delta^3\mu'},$$

and, therefore,

$$\begin{aligned} \Pi(\varphi\gamma, \mu\gamma) &= \Pi(\varphi'\gamma', \mu'\gamma') - \Pi(\varphi'\gamma', \rfloor\gamma' - \mu'\gamma') \\ &= \Pi(\varphi'\gamma', \mu'\gamma') + \Pi(\varphi'\gamma', \rfloor\gamma' + \mu'\gamma'), \end{aligned} \tag{16}$$

or, writing for brevity $\mu^{\backprime}$ for the co-amplitude of μ',

$$\Pi(\varphi\gamma, \mu\gamma) = \Pi(\varphi'\gamma', \mu'\gamma') - \Pi(\varphi'\gamma', \mu^{\backprime}\gamma'); \quad \mu'\gamma' + \mu^{\backprime}\gamma' = \rfloor\gamma'. \tag{16'}$$

This I call Somoff's theorem. Somoff, however, makes no use of it in this form. He employs a relation which is obtained from (16) by expressing $\Pi(\varphi'\gamma', \rfloor\gamma' - \mu'\gamma')$ in terms of $\Pi(\varphi'\gamma', \mu'\gamma')$ by (6). We have thus

$$\begin{aligned} \Pi(\varphi\gamma, \mu\gamma) &= 2\Pi(\varphi'\gamma', \mu'\gamma') - \gamma'^2\varphi'\gamma' \sin\mu' \frac{\cos\mu'}{\Delta\mu'} + \tfrac{1}{2}l\, \frac{\Delta(\mu'\gamma' - \varphi'\gamma') - \gamma'}{\Delta(\mu'\gamma' + \varphi'\gamma') - \gamma'} \\ &= 2\Pi(\varphi'\gamma', \mu'\gamma') - \gamma\varphi\gamma \sin\mu + \tfrac{1}{2}l\, \frac{1 + \gamma\sin\mu\sin\varphi}{1 - \gamma\sin\mu\sin\varphi}. \end{aligned} \tag{17}$$

[TO BE CONTINUED.]

A SIMPLE DISCUSSION OF LOGARITHMIC ERRORS.

By PROF. H. A. HOWE, Denver, Col.

[CONTINUED FROM VOL. II, PAGE 43.]

PROBLEM XI.

To find the probable error of an interpolated logarithmic trigonometric function.

Tables of logarithmic trigonometric functions may be divided into three classes. The first class embraces those in which the tabular difference is to be multiplied by a decimal of one, two, three, or, occasionally, four places. In this class we find 5, 6, and 7-place tables in which the functions are given for each 10 seconds or for each second, those tables in which the functions are given for each minute, and are interpolated for decimal parts of a minute, the 4-place tables in which the functions are given for each 10 minutes, and tables in which the differences of the successive values of the argument are tenths or hundredths of a degree.

The second class includes those tables in which the functions are given for